Countability

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ - infinite sets

cardinality of infinite sets

Recall that two sets A and B have same cardinality ($|A| = |B|$) iff there is a bijection from A to B.

example) $\mathbb{N}$ and $\mathbb{Z}$ have same cardinality

$f: \mathbb{N} \to \mathbb{Z}$ a bijection

$f(n) = n/2$ when n even $\to \{0, 1, 2, 3 \ldots\}$

$f(n) = -(n+1)/2$ when n odd. $\to \{-1, -2, -3 \ldots\}$

since f is a bijection, $|\mathbb{N}| = |\mathbb{Z}|$

An infinite set A is countably infinite iff there is a bijection from $\mathbb{N}$ to A.

$\mathbb{N}$ serve as an index.

Similarly ($\mathbb{Z}$ to A)

Finite sets are also countable (e.g., $[0, 4]$)

Cantor Schroeder Bernstein Theorem:

$|A| \leq |B|$ iff there exists a one-to-one function from A to B.

$$\left.\begin{aligned} &\text{if } f: A \to B \text{ one to one} \to |A| \leq |B| \\ &\text{if } g: B \to A \text{ one to one} \to |B| \leq |A| \end{aligned}\right\} \quad |A| = |B|$$

example) non-negative rationals are countably infinite.

$\rightarrow \quad |\mathbb{Q}^{\geq 0}| = |\mathbb{N}|$

$f: \mathbb{N} \rightarrow \mathbb{Q}^{\geq 0}$ $\qquad f(n) = n$ one to one

$g: \mathbb{Q}^{\geq 0} \rightarrow \mathbb{N}$ $\qquad g(\frac{x}{y}) \rightarrow$ write as (x, y) where $\frac{x}{y}$ is rational # in lowest terms.

$\rightarrow 2^x 3^y$ one to one

$\mathbb{Q}^{\geq 0}$ is countably infinite